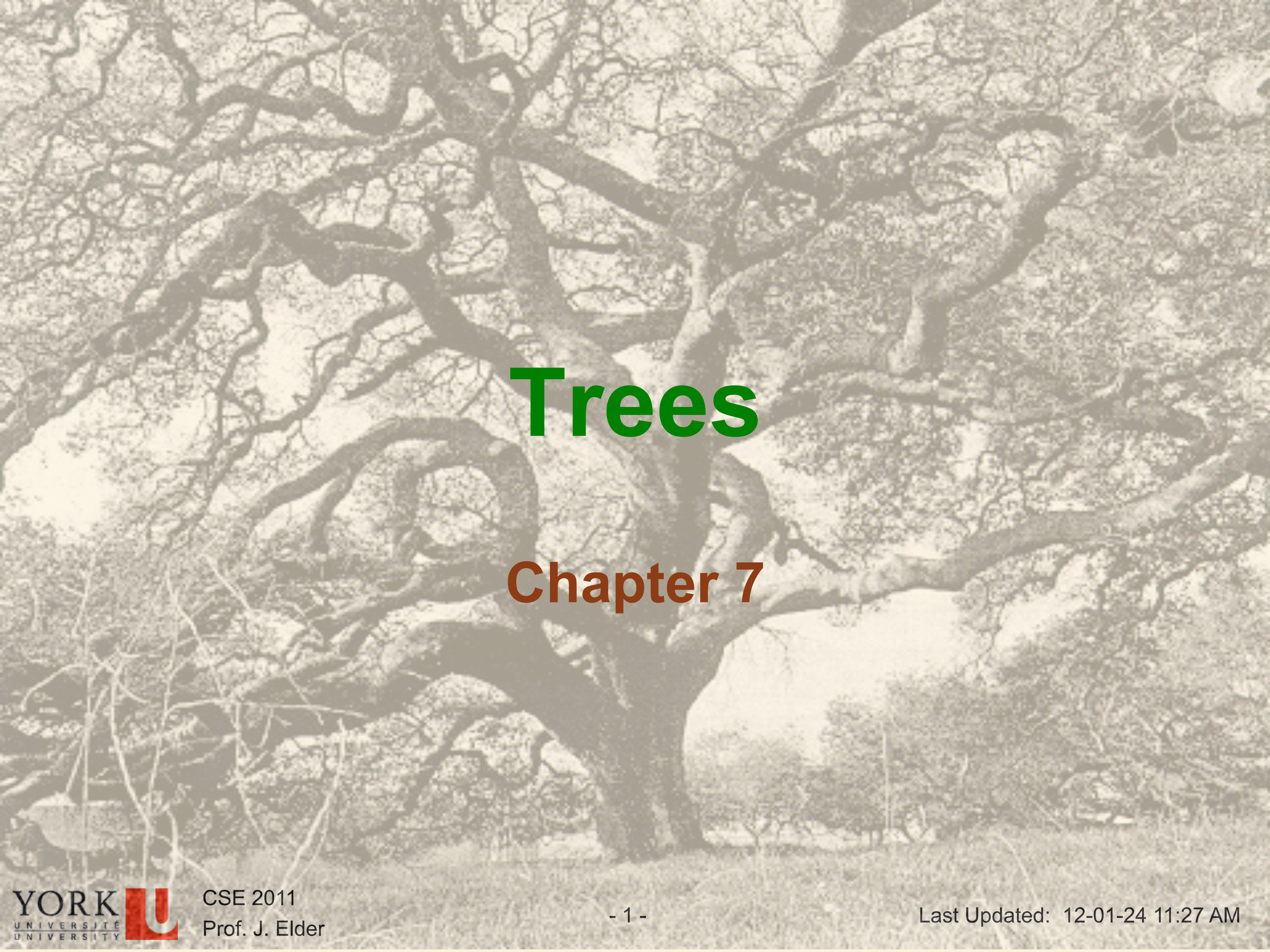
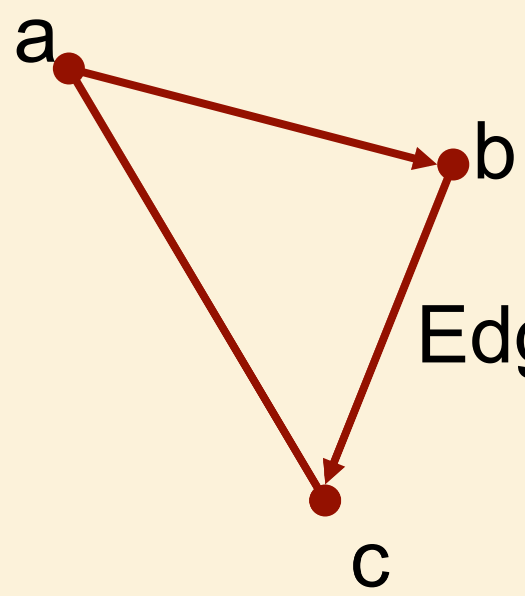


Trees

Chapter 7

Graph



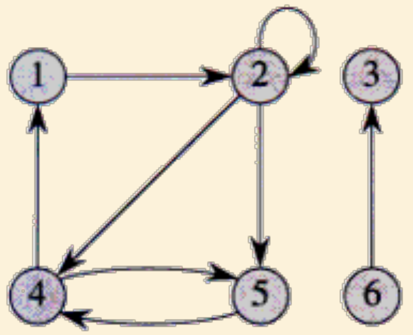
Node ~ city or computer

Edge ~ road or data cable

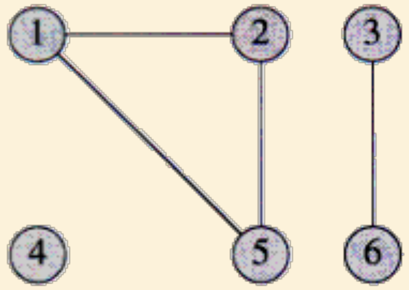
Undirected or Directed

A surprisingly large number of computational problems can be expressed as graph problems.

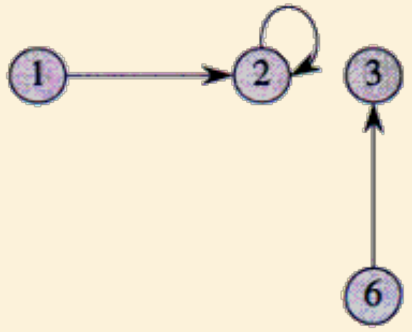
Directed and Undirected Graphs



(a)



(b)



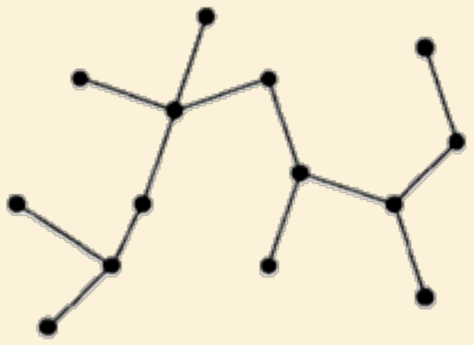
(c)

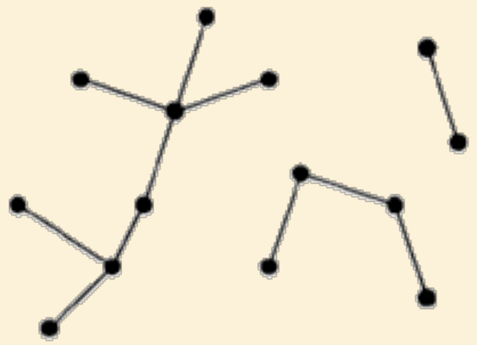
(a) A directed graph $G = (V, E)$, where $V = \{1,2,3,4,5,6\}$ and $E = \{(1,2), (2,2), (2,4), (2,5), (4,1), (4,5), (5,4), (6,3)\}$. The edge (2,2) is a self-loop.

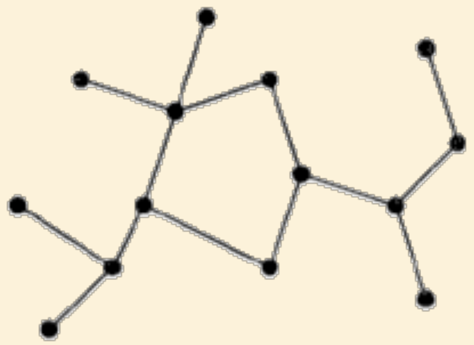
(b) An undirected graph $G = (V,E)$, where $V = \{1,2,3,4,5,6\}$ and $E = \{(1,2), (1,5), (2,5), (3,6)\}$. The vertex 4 is isolated.

(c) The subgraph of the graph in part (a) induced by the vertex set $\{1,2,3,6\}$.

Trees


Tree


Forest

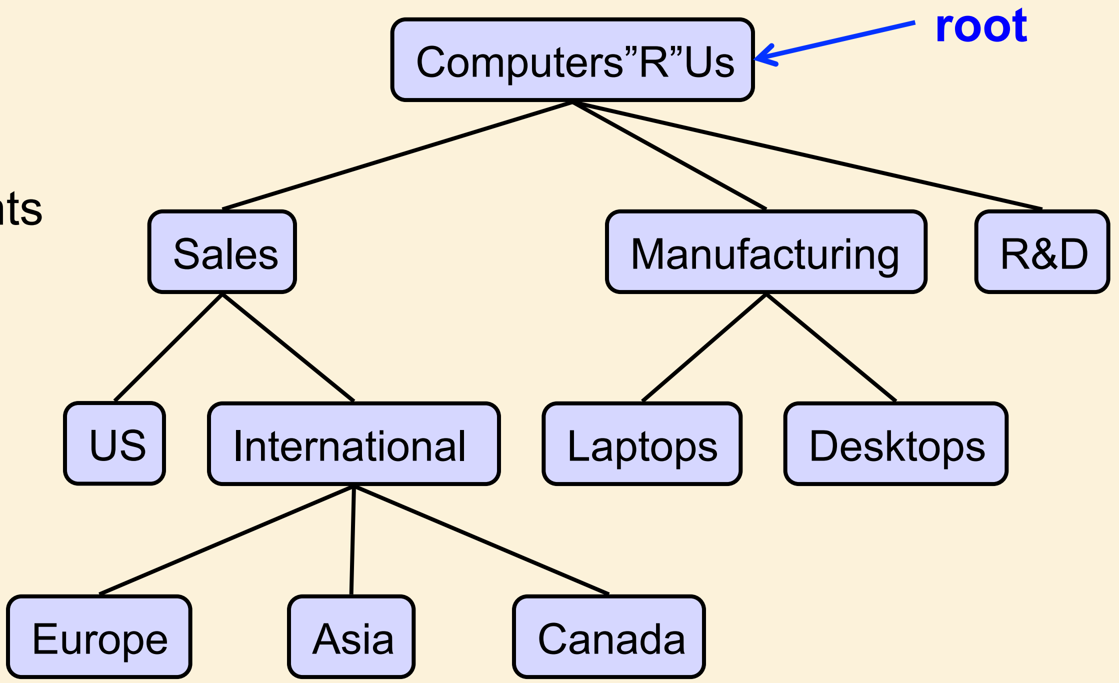

Graph with Cycle

A tree is a connected, acyclic, undirected graph.

A forest is a set of trees (not necessarily connected)

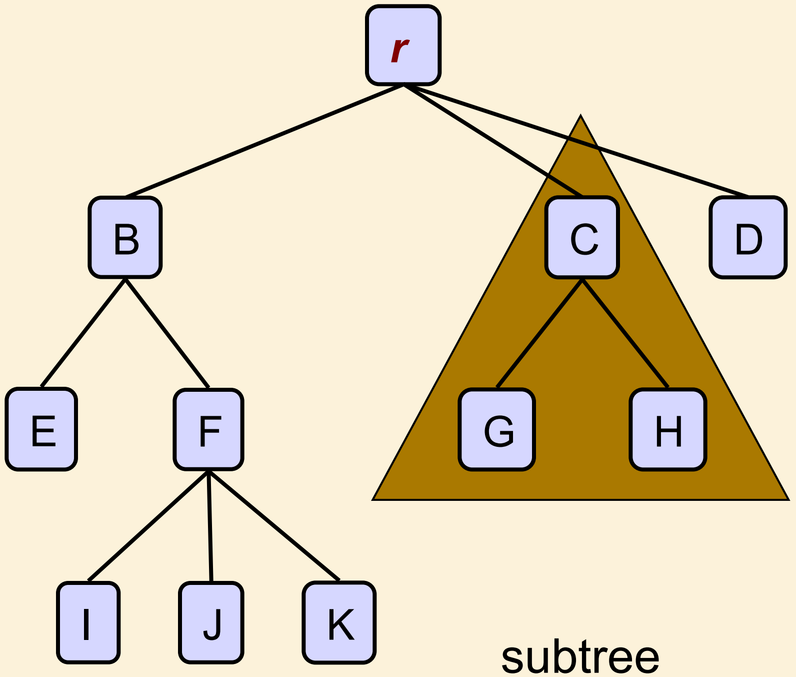
Rooted Trees

- Trees are often used to represent hierarchical structure
- In this view, one of the vertices (nodes) of the tree is distinguished as the root.
- This induces a parent-child relationship between nodes of the tree.
- Applications:
  - Organization charts
  - File systems
  - Programming environments



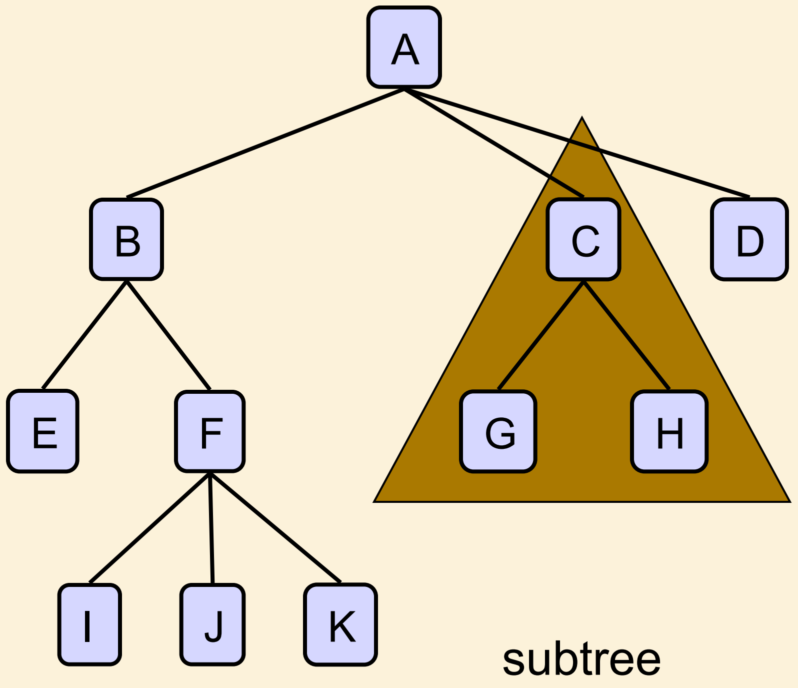
Formal Definition of Rooted Tree

- A rooted tree may be empty.
- Otherwise, it consists of
  - A root node ***r***
  - A set of **subtrees** whose roots are the children of ***r***



Tree Terminology

- **Root:** node without parent (A)
- **Internal node:** node with at least one child (A, B, C, F)
- **External node (a.k.a. leaf):** node without children (E, I, J, K, G, H, D)
- **Ancestors of a node:** self, parent, grandparent, great-grandparent, etc.
  - NB: A node is considered an ancestor of itself!
- **Descendent of a node:** self, child, grandchild, great-grandchild, etc.
  - NB: A node is considered a descendent of itself!
- **Siblings**: two nodes having the same parent
- **Depth of a node:** number of ancestors (excluding the node itself)
- **Height of a tree:** maximum depth of any node (3)
- **Subtree:** tree consisting of a node and its descendents



Traversing Trees

- One of the basic operations we require is to be able to traverse over the nodes of a tree.
- To do this, we will make use of a **Position ADT**.

Position ADT

- The Position ADT models the notion of place within a data structure where a single object is stored
- It gives a unified view of diverse ways of storing data, such as
  - a cell of an array
  - a node of a linked list
  - a node of a tree
- Just one method:
  - object **p.element()**: returns the element stored at the position **p**

Tree ADT

- We use positions to abstract the nodes of a tree.
- Generic methods:
  - integer size()
  - boolean isEmpty()
  - Iterator iterator()
  - Iterable positions()
- Accessor methods:
  - Position root()
  - Position parent(p)
  - Iterable children(p)
- Query methods:
  - boolean isInternal(p)
  - boolean isExternal(p)
  - boolean isRoot(p)
- Update method:
  - object replace(p, o)
  - Additional update methods may be defined by data structures implementing the Tree ADT

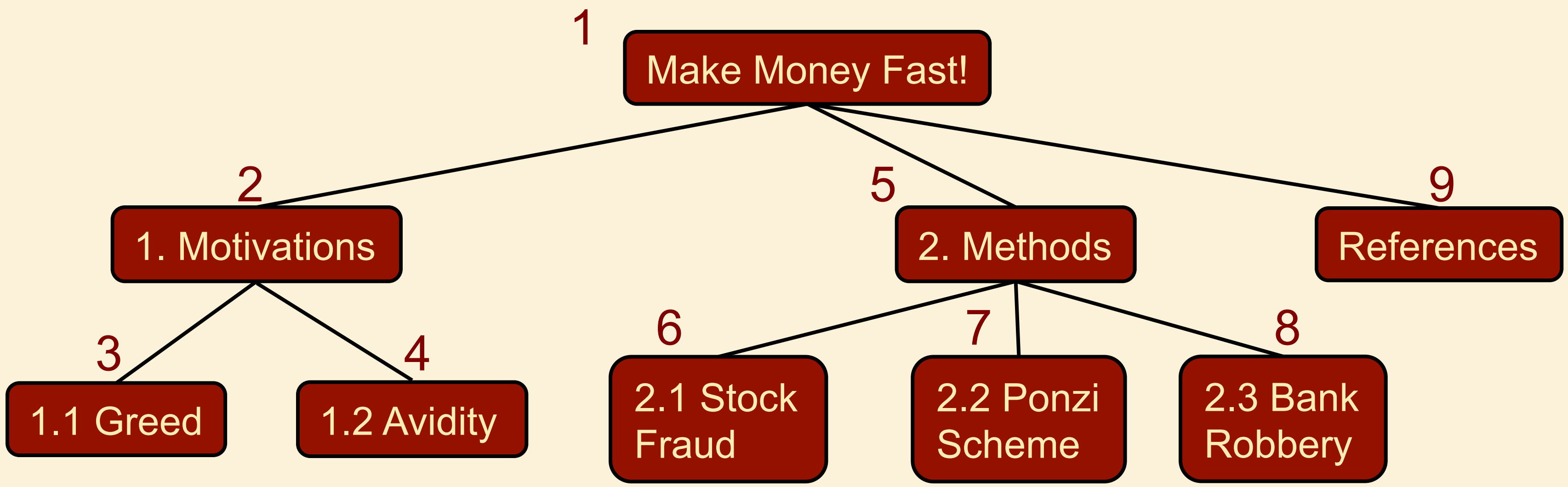
Positions vs Elements

- Why have both
  - Iterator iterator()
  - Iterable positions()
- The iterator returned by iterator() provides a means for stepping through the elements stored by the tree.
- The positions() method returns a collection of the nodes of the tree.
- Each node includes the element but also the links connecting the node to its parent and its children.

Preorder Traversal

- A traversal visits the nodes of a tree in a systematic manner
- Each time a node is visited, an action may be performed.
- Thus the order in which the nodes are visited is important.
- In a preorder traversal, a node is visited before its descendants

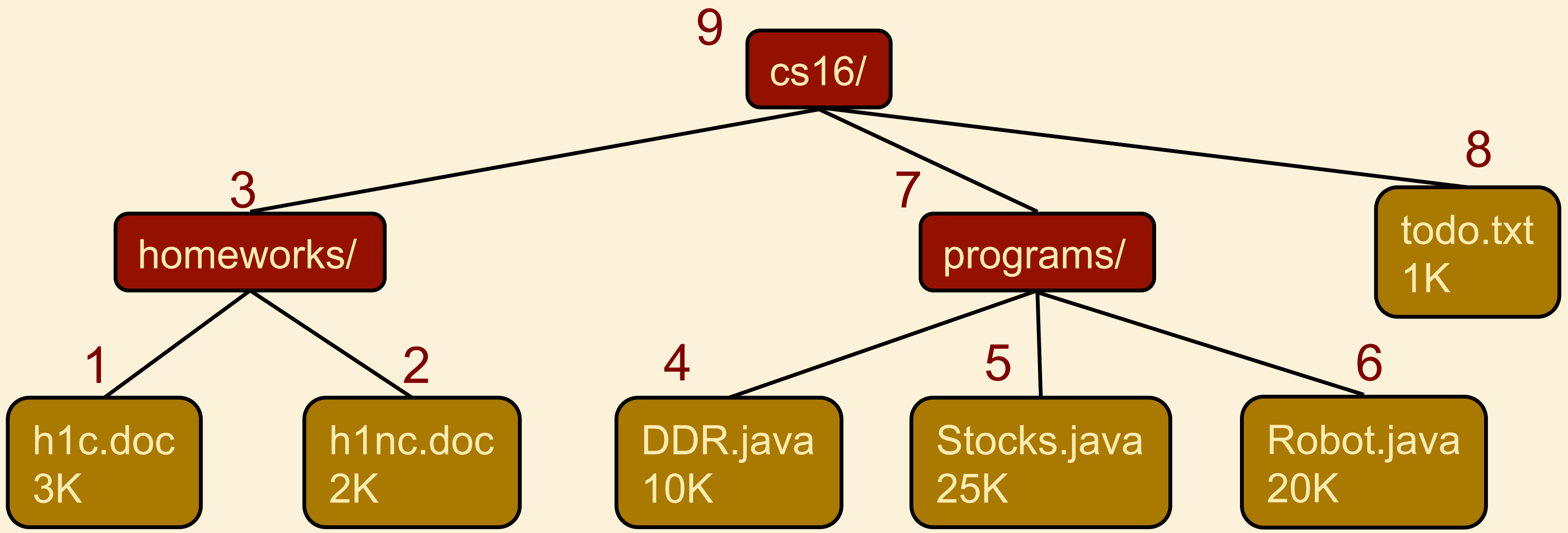
```
Algorithm preOrder(v)
    visit(v)
    for each child w of v
        preOrder (w)
```



Postorder Traversal

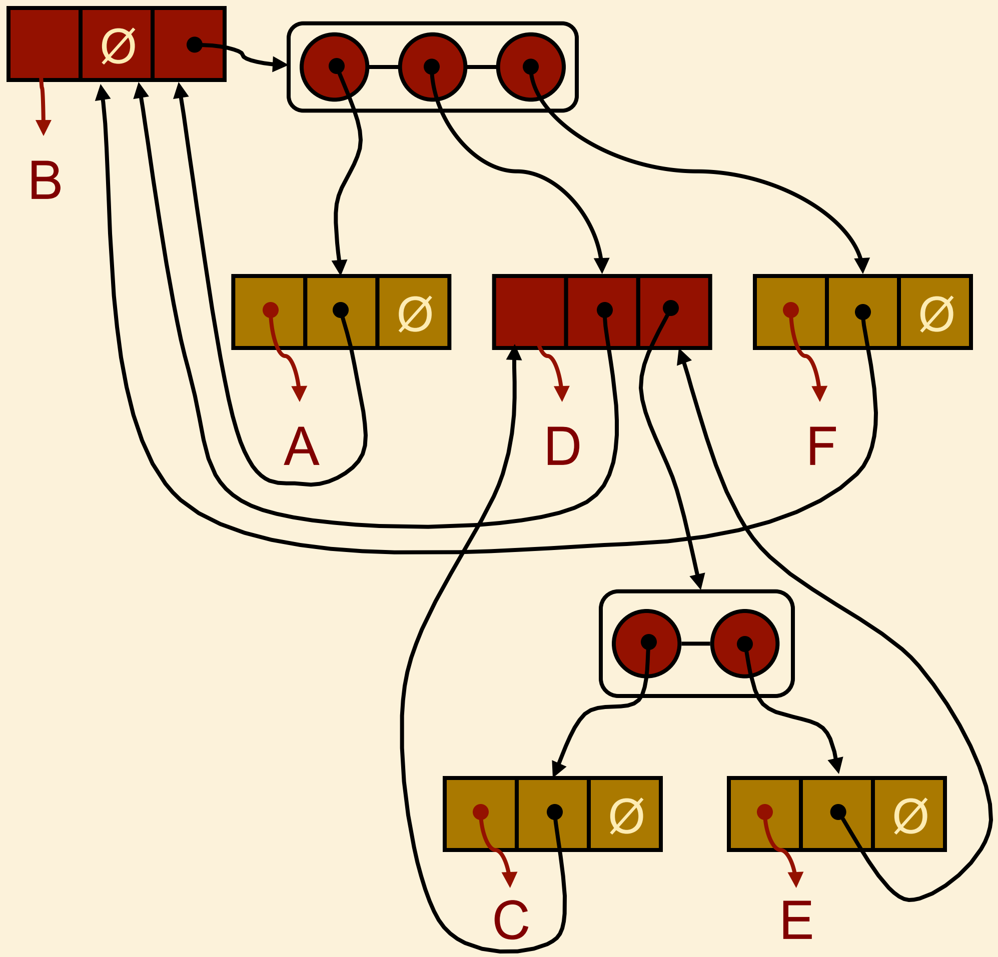
- In a postorder traversal, a node is visited after its descendants

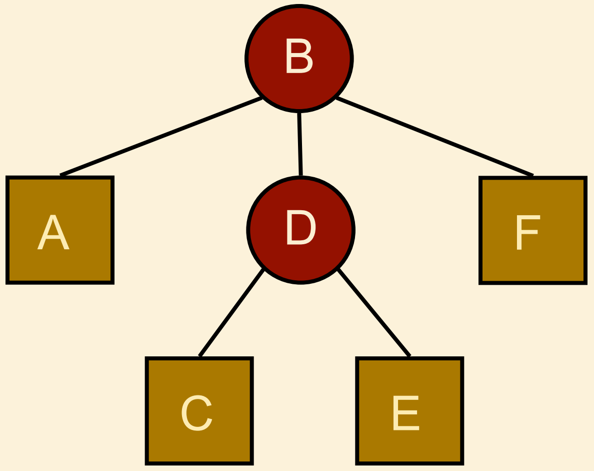
```
Algorithm postOrder(v)
    for each child w of v
        postOrder (w)
    visit(v)
```



Linked Structure for Trees

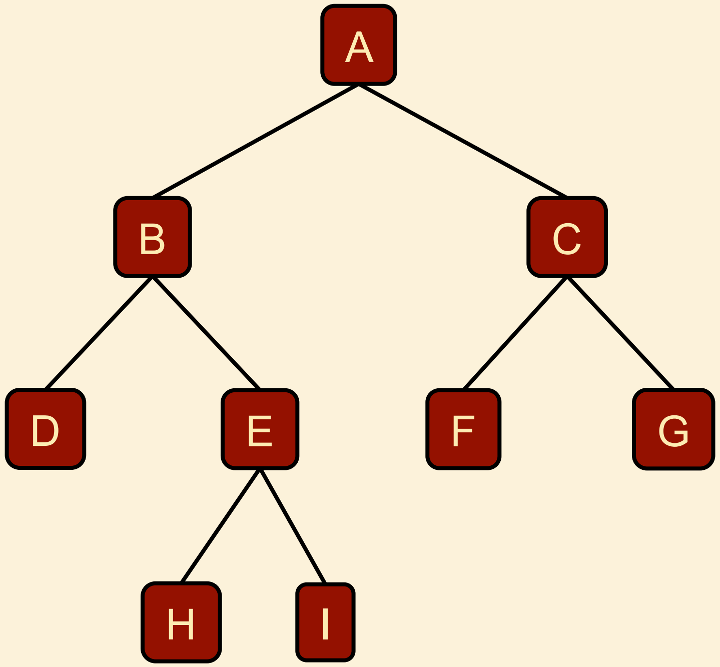
- A node is represented by an object storing
  - Element
  - Parent node
  - Sequence of children nodes
- Node objects implement the Position ADT





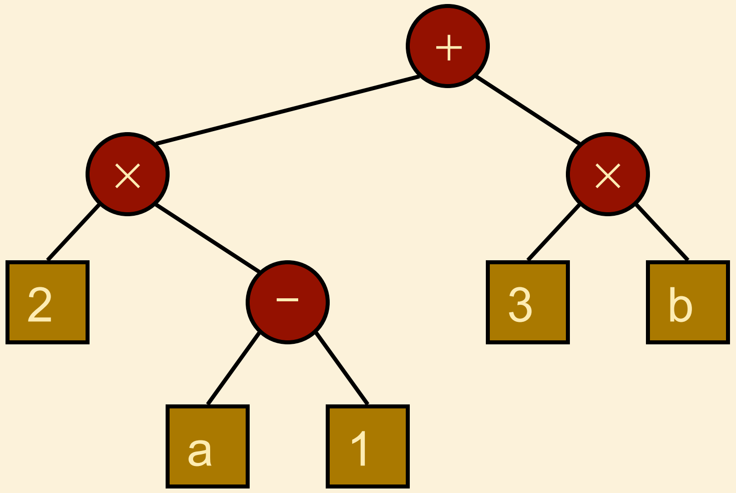
Binary Trees

- A **binary tree** is a tree with the following properties:
  - Each internal node has at most two children (exactly two for **proper** binary trees)
  - The children of a node are an ordered pair
- We call the children of an internal node **left child** and **right child**
- Applications:
  - arithmetic expressions
  - decision processes
  - searching



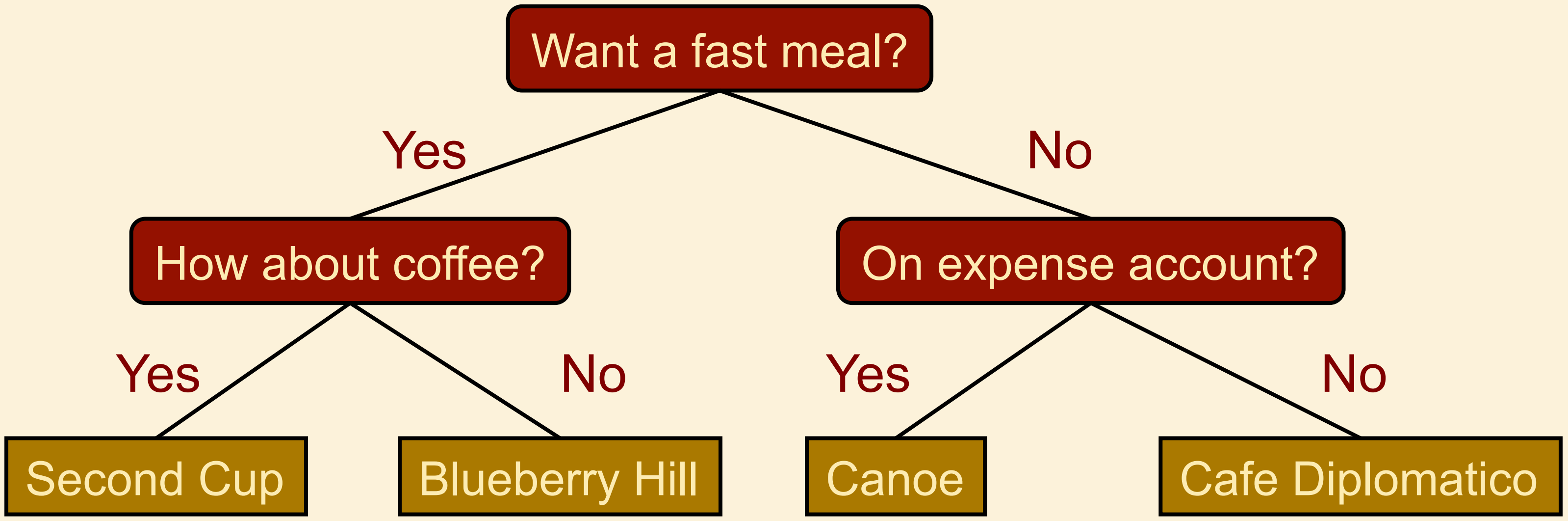
Arithmetic Expression Tree

- Binary tree associated with an arithmetic expression
  - internal nodes: operators
  - external nodes: operands
- Example: arithmetic expression tree for the expression $(2 \times (a - 1) + (3 \times b))$



Decision Tree

- Binary tree associated with a decision process
  - internal nodes: questions with yes/no answer
  - external nodes: decisions
- Example: dining decision



Proper Binary Trees

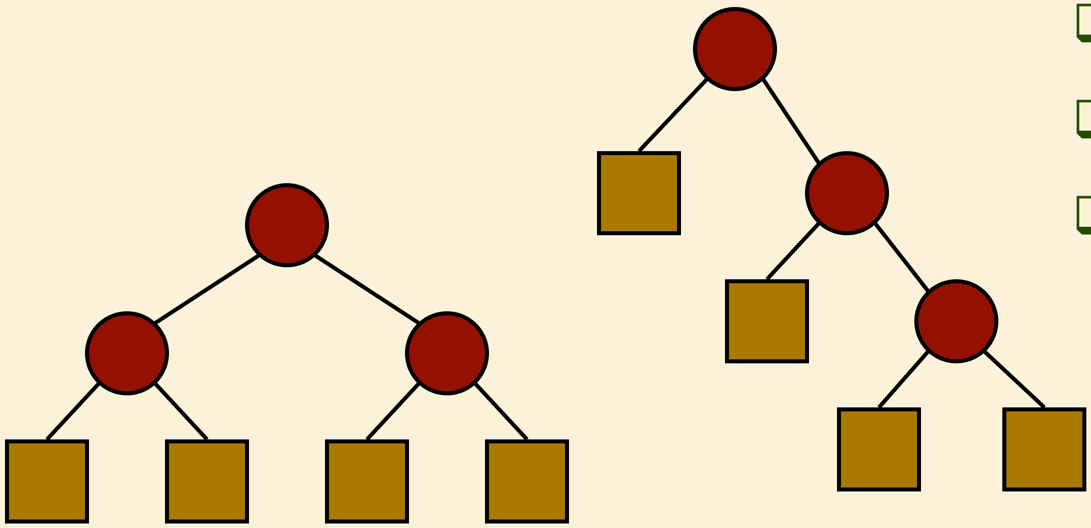
- A binary tree is said to be **proper** if each node has either 0 or 2 children.

Properties of Proper Binary Trees

- Notation

n	number of nodes
e	number of external nodes
i	number of internal nodes
h	height

- Properties:
  - $e = i + 1$
  - $n = 2e - 1$
  - $h \leq i$
  - $h \leq (n - 1)/2$
  - $e \leq 2^h$
  - $h \geq \log_2 e$
  - $h \geq \log_2(n + 1) - 1$



BinaryTree ADT

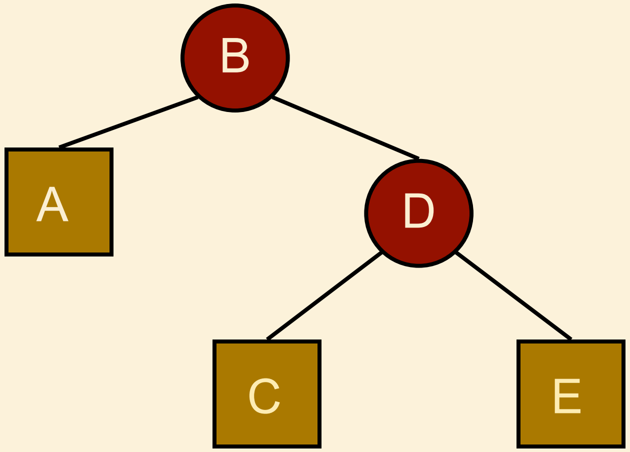
- The BinaryTree ADT extends the Tree ADT, i.e., it inherits all the methods of the Tree ADT
- Additional methods:
  - Position **left**(p)
  - Position **right**(p)
  - boolean **hasLeft**(p)
  - boolean **hasRight**(p)
- Update methods may be defined by data structures implementing the BinaryTree ADT

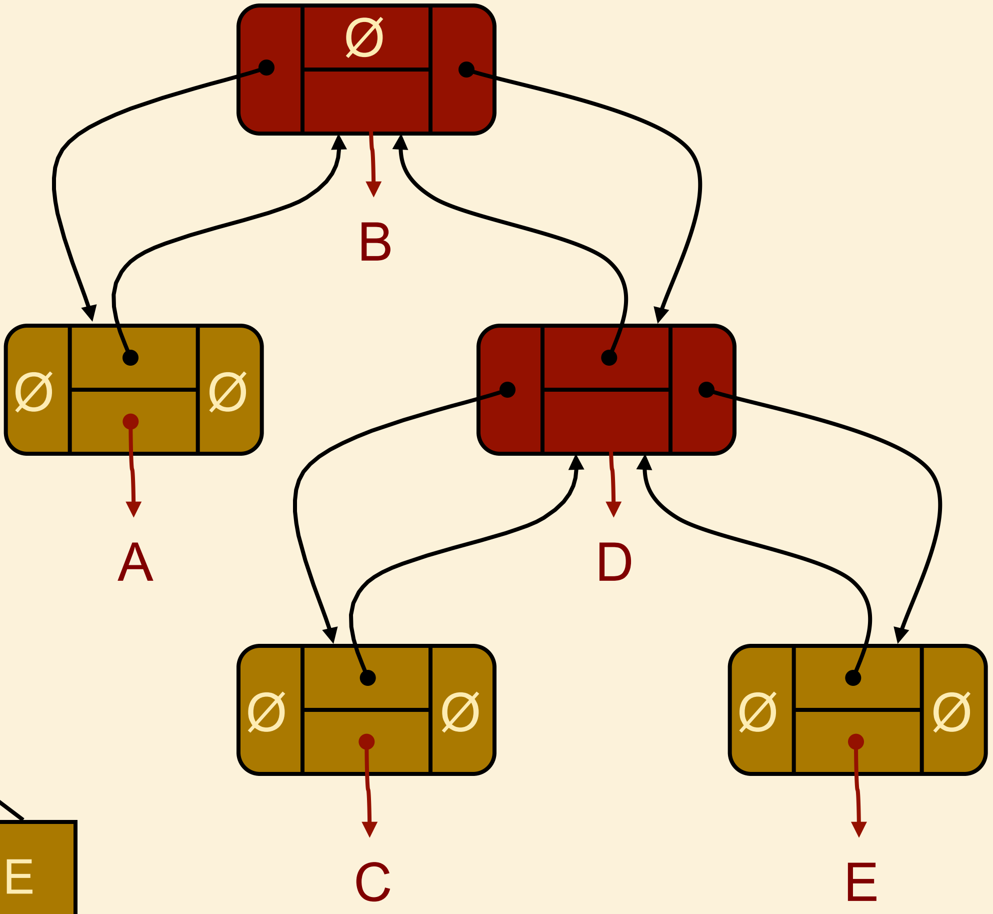
Representing Binary Trees

- Linked Structure Representation
- Array Representation

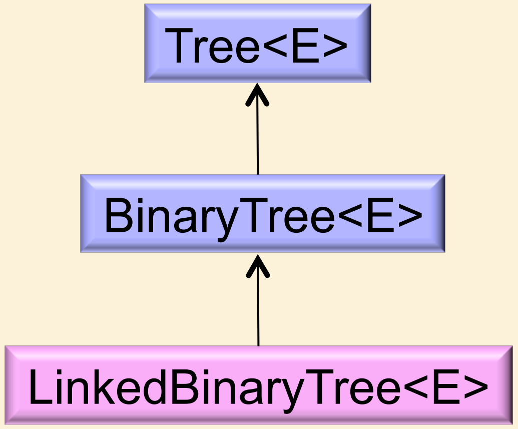
Linked Structure for Binary Trees

- A node is represented by an object storing
  - Element
  - Parent node
  - Left child node
  - Right child node
- Node objects implement the Position ADT





Implementation of Linked Binary Trees in net.datastructures



Query Methods:

- size()
- isEmpty()
- isInternal(p)
- isExternal(p)
- isRoot(p)
- hasLeft(p)
- hasRight(p)
- root()
- left(p)
- right(p)
- parent(p)
- children(p)
- sibling(p)
- positions()
- iterator()

Modification Methods:

- replace(p, e)
- addRoot(e)
- insertLeft(p)
- insertRight(p)
- remove(e)
- …

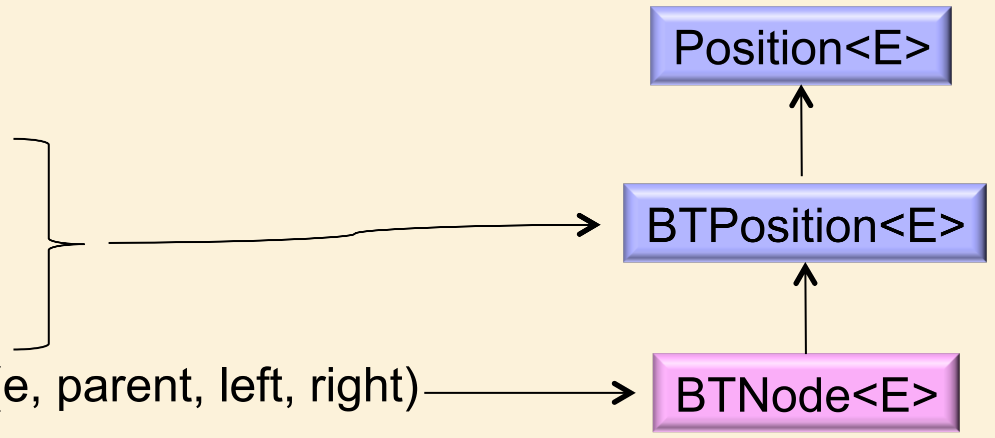
BTPosition in net.datastructures

- The implementation of Positions for binary trees in net.datastructures is a bit subtle.
  - BTPosition<E> is an interface in net.datastructures that represents the positions of a binary tree. This is used extensively to define the types of objects used by the LinkedBinaryTree<E> class that LinkedBinaryTreeLevel<E> extends.
  - You do not have to implement BTPosition<E>: it is already implemented by BTNode<E>. Note that LinkedBinaryTree<E> only actually uses the BTNode<E> class explicitly when instantiating a node of the tree, in method createNode. In all other methods it refers to the nodes of the tree using the BTPosition<E> class (i.e., a widening cast). This layer of abstraction makes it easier to change the specific implementation of a node down the road – it would only require a change to the one method createNode.
  - We use BTPosition<E> in testLinkedBinary to define the type of father, mother, daughter and son, and to cast the returned values from T.addRoot, T.insertLeft and T.insertRight. These three methods are implemented by LinkedBinaryTree and return nodes created by the createNode method.
  - In LinkedBinaryTreeLevel, you can use the BTPosition<E> interface to define the type of the nodes stored in your NodeQueue. These nodes will be returned from queries on your binary tree, and thus will have been created by the createNode method using the BTNode<E> Class.

LinkedBinaryTree

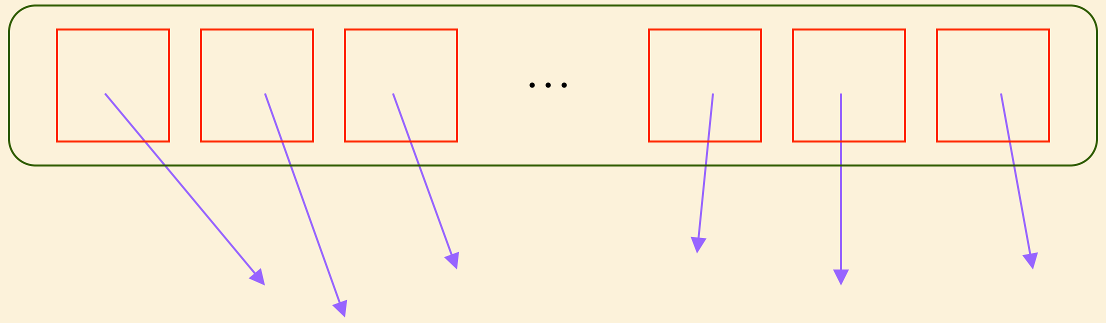
- public hasLeft (p)
- public hasRight (p)
- …

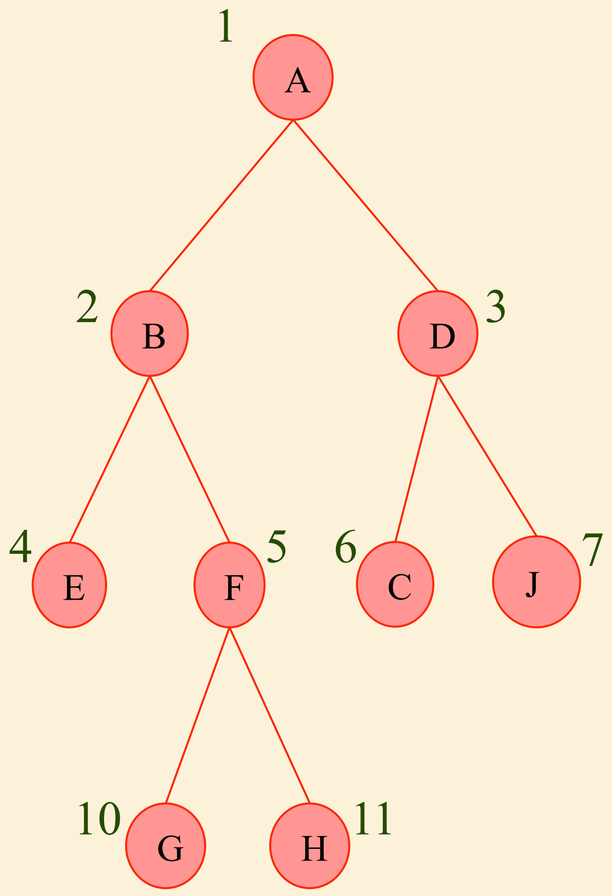
- protected createNode (e, parent, left, right)



Array-Based Representation of Binary Trees

- nodes are stored in an array, using a **level-numbering** scheme.





- let rank(node) be defined as follows:
  - rank(root) = 1
  - if node is the **left** child of parent(node), rank(node) = **2*rank(parent(node))**
  - if node is the **right** child of parent(node), rank(node) = **2*rank(parent(node))+1**

Comparison

Linked Structure

- Requires explicit representation of 3 links per position:
  - parent, left child, right child
- Data structure grows as needed – no wasted space.

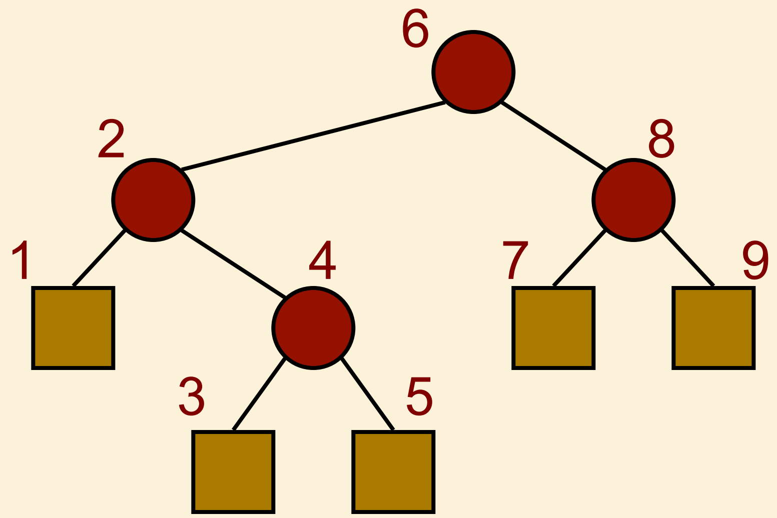
Array

- Parent and children are implicitly represented:
  - Lower memory requirements per position
- Memory requirements determined by height of tree. If tree is **sparse**, this is highly inefficient.

Inorder Traversal of Binary Trees

- In an inorder traversal a node is visited after its left subtree and before its right subtree
- Application: draw a binary tree
  - x(v) = inorder rank of v
  - y(v) = depth of v

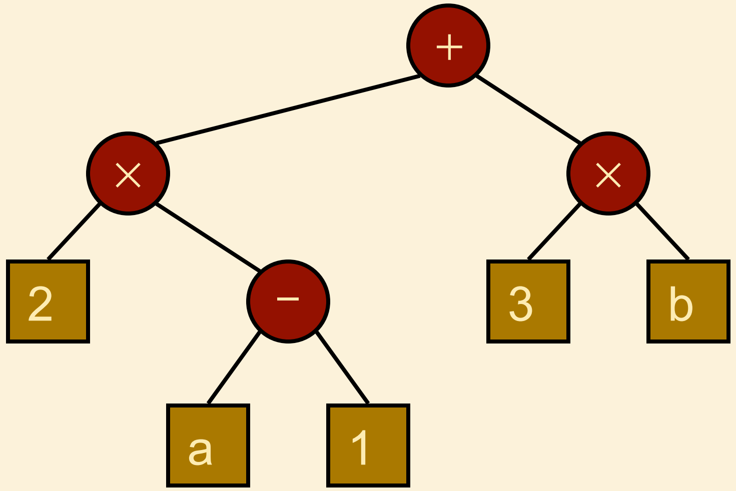
```
Algorithm inOrder(v)
    if hasLeft (v)
        inOrder (left (v))
    visit(v)
    if hasRight (v)
        inOrder (right (v))
```



Print Arithmetic Expressions

- Specialization of an inorder traversal
  - print operand or operator when visiting node
  - print "(" before traversing left subtree
  - print ")" after traversing right subtree

Input:



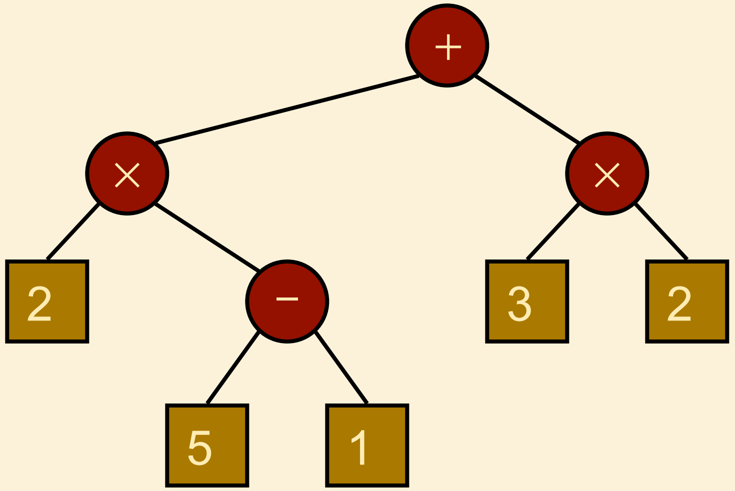
```
Algorithm printExpression(v)
    if hasLeft (v)
        print("(")
        inOrder (left(v))
    print(v.element ())
    if hasRight (v)
        inOrder (right(v))
        print (")")
```

Output:

$((2 \times (a - 1)) + (3 \times b))$

Evaluate Arithmetic Expressions

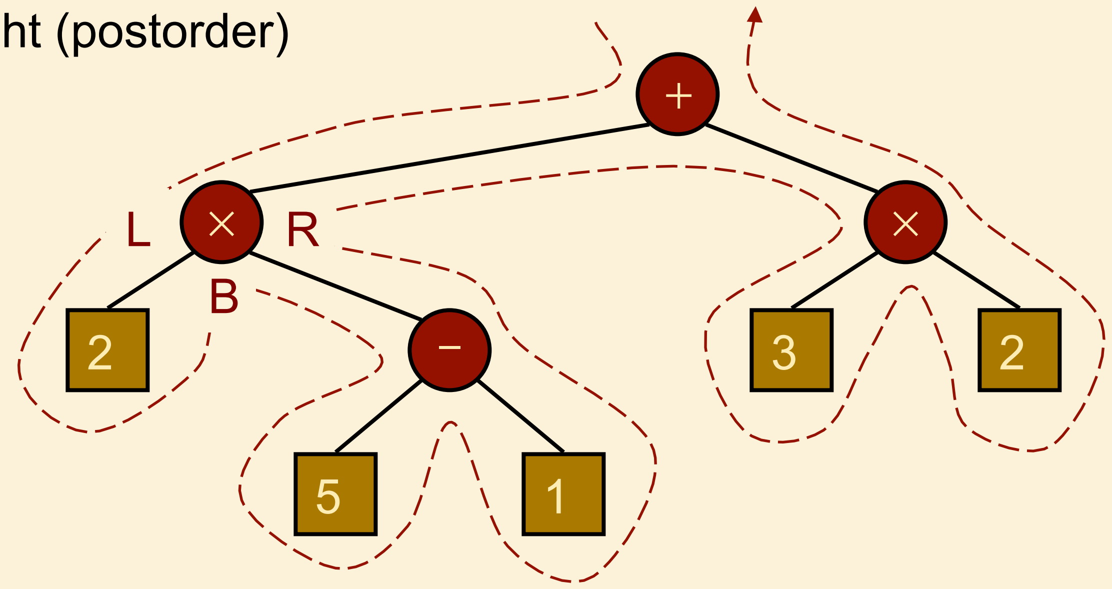
- Specialization of a postorder traversal
  - recursive method returning the value of a subtree
  - when visiting an internal node, combine the values of the subtrees



```
Algorithm evalExpr(v)
    if isExternal (v)
        return v.element ()
    else
        x ← evalExpr (leftChild (v))
        y ← evalExpr (rightChild (v))
        ◆ ← operator stored at v
        return x ◆ y
```

Euler Tour Traversal

- Generic traversal of a binary tree
- Includes as special cases the preorder, postorder and inorder traversals
- Walk around the tree and visit each node three times:
  - on the left (preorder)
  - from below (inorder)
  - on the right (postorder)



Template Method Pattern

- Generic algorithm that can be specialized by redefining certain steps
- Implemented by means of an abstract Java class
- Visit methods can be redefined by subclasses
- Template method eulerTour
  - Recursively called on the left and right children
  - A Result object with fields leftResult, rightResult and finalResult keeps track of the output of the recursive calls to eulerTour

```
public abstract class EulerTour {
    protected BinaryTree tree;
    protected void visitExternal(Position p, Result r) { }
    protected void visitLeft(Position p, Result r) { }
    protected void visitBelow(Position p, Result r) { }
    protected void visitRight(Position p, Result r) { }
    protected Object eulerTour(Position p) {
        Result r = new Result();
        if tree.isExternal(p) { visitExternal(p, r); }
            else {
                visitLeft(p, r);
                r.leftResult = eulerTour(tree.left(p));
                visitBelow(p, r);
                r.rightResult = eulerTour(tree.right(p));
                visitRight(p, r);
                return r.finalResult;
            } …
```

Specializations of EulerTour

- We show how to specialize class EulerTour to evaluate an arithmetic expression
- Assumptions
  - External nodes store Integer objects
  - Internal nodes store Operator objects supporting method

    operation (Integer, Integer)

```
public class EvaluateExpression
                        extends EulerTour {

    protected void visitExternal(Position p, Result r) {
        r.finalResult = (Integer) p.element();
    }

    protected void visitRight(Position p, Result r) {
        Operator op = (Operator) p.element();
        r.finalResult = op.operation(
                            (Integer) r.leftResult,
                            (Integer) r.rightResult
                            );
    }

    …
}
```